Journal of Engineering Mathematics **34:** 319–334, 1998.

The dispersion relation and eigenfunction expansions for water waves in a porous structure

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Received 16 January 1998; accepted in revised form 18 May 1998

Abstract. The Sollitt-and-Cross model of water-wave motion in a porous structure involves a free-surface condition which contains a complex parameter. This leads to two particular difficulties when this model is used in conjunction with eigenfunction expansion techniques. First of all the roots of the dispersion relation are themselves complex and therefore difficult to locate by standard numerical methods. Secondly, the vertical eigenfunction problem is not self-adjoint and standard expansion theorems do not apply. In this paper it is shown how these two difficulties may be resolved with the aid of the theories of, respectively, complex variables and non-self-adjoint differential operators. In particular, a method is described that allows the explicit calculation of the roots of the dispersion relation, and the appropriate expansion theorem is given.

Keywords: dispersion relation, eigenfunctions, expansion theorem, non-self-adjoint, separation of variables.

1. Introduction

A number of authors have investigated the interaction of water waves with porous structures, such as rubble-mound breakwaters, using the linearised theory of Sollitt and Cross [1]. A fundamental aspect of the model is that, within a surface-piercing structure, the free-surface condition has the same form as that for the standard linearised water-wave problem, but the frequency parameter is complex rather than real. There are two main approaches to the application of this theory to practical problems, namely boundary-element methods [*e.g.* 2] and eigenfunction expansion methods [*e.g.* 3]. The latter method is the subject of the present paper.

In the eigenfunction approach the complex parameter in the free-surface conditon leads to two particular difficulties. Firstly, the roots of the dispersion relation are themselves complex and are therefore more difficult to locate than in the standard water-wave problem where they are either purely real or purely imaginary. Here results are given that enable the straightforward evaluation of these roots of the dispersion relation. The second difficulty is that the problem for the vertical eigenfunctions is no longer self-adjoint and the standard expansion theorems that appear in most text books do not apply. Here the problem is set into the context of the general theory of non-self-adjoint ordinary differential operators and the relevant expansion theorem is given.

Some relevant aspects of the linearised water-wave problem for time-harmonic motion of angular frequency ω in a region of constant depth h are now summarised. Cartesian coordinates (x, y, z) are chosen so that the vertical coordinate y has its origin in the free surface and is directed upwards. In the Sollitt-and-Cross model, the equation of motion includes resistance forces, which are quadratic in the velocity, and the added resistance due to the added mass of the individual grains within the medium. Under the assumption of time-periodic motion, this nonlinear equation may be linearised by an application of the principle of equivalent work. After the introduction of a velocity potential $\Phi(x, y, z, t)$, which by mass conservation satisfies Laplace's equation, the equation of motion may be integrated to obtain a Bernoulli equation

$$s\frac{\partial\Phi}{\partial t} + \frac{p}{\rho_W} + gy + \nu\omega\Phi = 0. \tag{1}$$

Here t is time, p is the pore pressure, ρ_W is the fluid density, g is the acceleration due to gravity, ν is a dimensionless friction coefficient and the inertia coefficient

$$s = 1 + \frac{1-\varepsilon}{\varepsilon}C_M, \tag{2}$$

where ε is the porosity of the medium and C_M is the added mass of the grains. On the fluid free surface within the porous medium, the Bernoulli equation (1) may be combined with the kinematic condition that water particles in the surface move with the surface, to obtain

$$s\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial y} + \nu\omega\frac{\partial\Phi}{\partial t} = 0 \quad \text{on} \;\; y = 0. \tag{3}$$

Further details of the model may be found elsewhere [1, 3].

If separable solutions are sought in the form $\Phi(x, y, z, t) = \mathrm{Re}\{\phi(x, z)\chi(y)\,\mathrm{e}^{-i\omega t}\}$, then it is found that the Laplace equation yields

$$T\chi \equiv -\frac{\mathrm{d}^2\chi}{\mathrm{d}y^2} = \lambda\chi \quad \text{for} \;\; -h < y < 0, \tag{4}$$

where λ is the separation constant, while the free surface condition (3) reduces to

$$\frac{\mathrm{d}\chi}{\mathrm{d}y} = K\chi \quad \text{on} \;\; y = 0, \tag{5}$$

where $K = \omega^2(s+i\nu)/g$. In addition, χ is required to satisfy the condition of no flow through the bed which is

$$\frac{\mathrm{d}\chi}{\mathrm{d}y} = 0 \quad \text{on} \;\; y = -h. \tag{6}$$

When the friction coefficient $\nu = 0$ and the porosity $\varepsilon = 1$, the above problem for χ reduces to that which arises in the standard water-wave problem when a series solution in terms of vertical eigenfunctions is sought by the method of separation of variables [4, Section 7.4].

First of all, the solution of (4–6) is considered for the standard water-wave problem in which K is the real number ω^2/g. The solutions of the problem are of the form

$$\chi = \cos k(y+h), \tag{7}$$

where $k = \lambda^{1/2}$ is a root of the dispersion relation

$$f(kh) \equiv Kh + kh\tan kh = 0. \tag{8}$$

For any specified value of the frequency parameter K, this dispersion relation has two purely imaginary roots $k = \pm k_0$ and an infinity of purely real roots $\{k = \pm k_m; m = 1, 2, \dots\}$. The set of vertical eigenfunctions

$$\chi_m = \frac{\cos k_m(y+h)}{N_m}, \quad m = 0, 1, 2, \dots, \tag{9}$$

with

$$N_m^2 = \frac{1}{2}\left(1 + \frac{\sin 2k_m h}{2k_m h}\right), \tag{10}$$

form a complete orthonormal set satisfying

$$\frac{1}{h}\int_{-h}^{0} \chi_m(y)\chi_n(y)\,\mathrm{d}y = \delta_{mn}, \tag{11}$$

where δ_{mn} is the Kronecker delta.

It is convenient to introduce an inner product notation. For any $u, v \in L^2(-h, 0)$, the space of complex-valued functions that are square-integrable over the depth, their inner product is defined by

$$\langle u, v\rangle = \frac{1}{h}\int_{-h}^{0} u\bar{v}\,\mathrm{d}y, \tag{12}$$

where the over bar denotes complex conjugate. In this notation, the orthogonality condition (11) is

$$\langle \chi_m, \chi_n\rangle = \delta_{mn}. \tag{13}$$

By the expansion theorem for self-adjoint problems [5, p. 199], for any $f \in L^2(-h, 0)$

$$f = \sum_{m=0}^{\infty}\langle f, \chi_m\rangle\chi_m. \tag{14}$$

As noted above, the Sollitt-and-Cross model [1] for time-harmonic fluid motion in a porous structure leads again to the consideration of the boundary-value problem (4–6), but with K now a complex number. It is numerically difficult to locate the roots of the dispersion relation in the complex plane. As noted above, for the case of real K the roots lie on either the real or imaginary axis in the complex k plane and are therefore easily located by standard numerical methods. For complex K the roots do not lie on the axes and standard iteration schemes may not converge to the required root due to the lack of a good initial guess. Dalrymple *et al*. [3] used a numerical scheme in which the roots are tracked individually as the imaginary part of K is incremented from zero. In general, it is not possible to make direct calculations for a specific K using this method. In contrast to this, the results given in Sections 2 and 3 of the present paper allow roots of the dispersion relation to be found explicitly for arbitrary K. Since the initial reviews of this paper it has come to the author's attention that some of the material presented here in Sections 2 and 3 overlaps with work carried out by Hazard and Lenoir [6, Appendix B].

A feature of the problem highlighted by Dalrymple *et al*. [3] is that for isolated values of the complex parameter K there are double roots of the dispersion relation (8). These double roots occur at the zeros of $f'(kh)$ and therefore are the roots of

$$\sin 2kh + 2kh = 0. \tag{15}$$

Other than $kh = 0$, all roots of (15) are complex. Again, it is possible to obtain explicit expressions for the complex roots and this is done here in Section 4.

Once a root of (15) has been located the corresponding value of K follows from (8). For such a value of K, the eigenfunctions (9) no longer form a complete set. Dalrymple *et al.* [3] obtain the missing eigenfunctions by an indirect argument based on the Green's function for the particular water-wave problem under consideration. Here, the problem is re-examined from the point of view of the general theory of non-self-adjoint ordinary differential operators and the formal expansion theorem is given in Section 5. Finally, in Section 6, the expansion theorem is illustrated by obtaining solutions to Laplace's equation in a two-dimensional problem.

2. Analysis of the dispersion relation

In this section the dispersion relation (8) is studied for complex K and a number of results are obtained concerning the location of roots in the complex plane. It is convenient to write

$$Kh = \Gamma \equiv \alpha + i\beta, \tag{16}$$

where α and β are both real, and $kh = w$ so that the dispersion relation is

$$f(w) \equiv \Gamma + w \tan w = 0 \tag{17}$$

and $w = u + iv$ is a complex variable. A simple observation is that if $w = w_0$ is a root then $w = -w_0$ is also a root. Within the context of the water-wave problem under discussion $\alpha \geqslant 0$ and this restriction will be adopted throughout this paper. It will also be assumed that $\beta > 0$, roots of the dispersion relation for $\beta < 0$ follow simply by taking the complex conjugate of the roots obtained for $\beta > 0$.

To give some appreciation of the numerical distribution of the roots, the zero contours of the real and imaginary parts of $f(w)$ are plotted in Figure 1. The filled dots are placed at $w = \pm(2n+1)\pi/2$, n an integer, which are the zeros of $\cos w$ at which $f(w)$ is undefined (we could avoid this problem by plotting contours of $f(w)\cos w$, but then the plots contain much more detail and are less clear). Roots of the dispersion relation correspond to the intersections of the two sets of contours but excluding those at $w = \pm(2n+1)\pi/2$. In the figure, α is fixed while β is varied. There is a double root of the dispersion relation at $w \approx \pm(2{\cdot}1062 - 1{\cdot}1254i)$ for $\alpha \approx 1{\cdot}6506$, $\beta \approx 2{\cdot}05995$; Figure 1 gives plots for this α and as β is varied it can be seen how the double root corresponds to a 'pinching off' of the zero contour for the imaginary part of $f(w)$.

Useful information about the location of the roots can be obtained from the argument principle of complex variable theory [7, Sect. 12]. For a function g of the complex variable w that is analytic within a closed contour C, the argument principle states that the increment in the argument of the complex number $g(w)$ as C is traversed in the positive (that is anticlockwise)

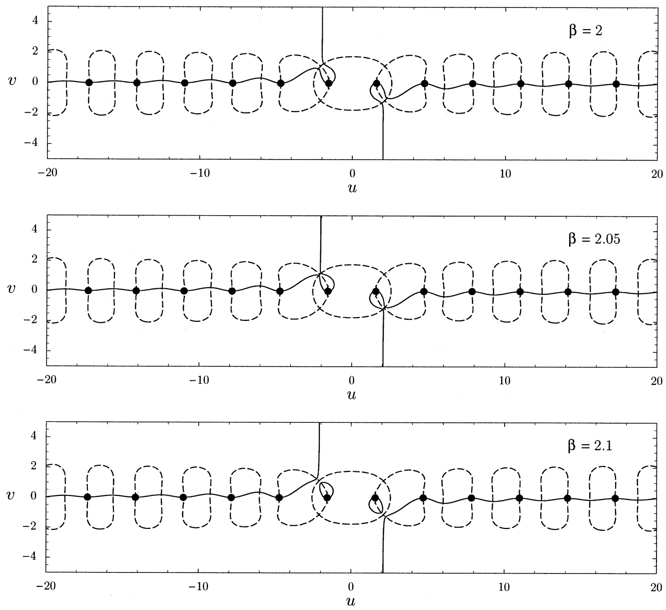

Figure 1. Zero contours of the real (– – – –) and imaginary (———) parts of $\alpha + i\beta + (u + iv)\tan(u + iv)$ for $\alpha = 1.6506$.

direction, here denoted by $[\arg g]_C$, is equal to 2π times the number of zeros of g within C. The argument of $g(w)$ must be regarded as varying continuously as C is traversed and so is not restricted to lie within an interval of length 2π.

To apply the argument principle, the dispersion relation (17) is first rewritten in the form

$$g(w) \equiv \Gamma \cos w + w \sin w = 0, \tag{18}$$

(note that g and f have the same zeros). Guided by the periodicity of the trigonometric functions and the numerical results shown in Figure 1, we choose C to be the rectangle centered on $(u, v) = (n\pi, 0)$ and passing through the four points $(u, v) = (R_n, \pm S)$ and $(u, v) = (R_{n-1}, \pm S)$, where $R_n = (2n + 1)\pi/2$, so that

$$C = C_1 \cup C_2 \cup C_3 \cup C_4, \tag{19}$$

where

$$\begin{aligned} &C_1 = \{u = R_n, v \in (-S, S]\}, \qquad C_2 = \{v = S, u \in [R_{n-1}, R_n)\}, \\ &C_3 = \{u = R_{n-1}, v \in [-S, S)\}, \qquad C_4 = \{v = -S, u \in (R_{n-1}, R_n]\}, \end{aligned} \tag{20}$$

and the limit $S \to \infty$ will be taken eventually.

On C_1,

$$g(w) = g_1(v) \equiv (-1)^n h_1(v), \tag{21}$$

where

$$h_1(v) = \{\beta \sinh v + R_n \cosh v + i(-\alpha \sinh v + v \cosh v)\}. \tag{22}$$

It is simpler to consider the function $h_1(v)$ which has the same argument as $g_1(v)$. The aim is to calculate the change in the argument of h_1 as C_1 is traversed, that is as v increases from negative to positive infinity. First of all note that

$$\arg h_1(v) \to \begin{cases} \frac{1}{2}\pi \,(\mathrm{mod}\, 2\pi) & \text{as } v \to \infty, \\ -\frac{1}{2}\pi \,(\mathrm{mod}\, 2\pi) & \text{as } v \to -\infty. \end{cases} \tag{23}$$

More specifically, as $v \to \infty$

$$h_1(v) \sim \tfrac{1}{2}\,\mathrm{e}^{v}\{\beta + R_n + i(-\alpha + v)\}, \tag{24}$$

so that $h_1(v)$ lies asymptotically in the first quadrant of the complex h_1 plane as C_1 is traversed. The argument of $h_1(v)$ must vary continuously and the increase in this argument will depend on if, and where, the imaginary h_1 axis is crossed. From (22), $\mathrm{Re}\, h_1(v) = 0$ at $v = -\tanh^{-1} R_n/\beta$. Clearly, there are no real solutions, and hence no crossings of the imaginary axis, for $\beta < |R_n|$ and then $[\arg h_1]_{C_1} = [\arg g_1]_{C_1} = \pi$. If $\beta > |R_n|$, so that the contour in the h_1 plane does cross the $\mathrm{Im}\, h_1$ axis, then from (22) the only crossing occurs at

$$\mathrm{Im}\, h_1(v) = \left(\frac{\alpha R_n}{\beta} - \tanh^{-1}\frac{R_n}{\beta}\right)\cosh v. \tag{25}$$

If $\tanh(\alpha R_n/\beta) < R_n/\beta$ the crossing occurs in the lower half-plane and again $[\arg h_1]_{C_1} = [\arg g_1]_{C_1} = \pi$. If $\tanh(\alpha R_n/\beta) > R_n/\beta$ the crossing occurs in the upper half-plane and then $[\arg h_1]_{C_1} = [\arg g_1]_{C_1} = -\pi$.

Similar results apply as C_3 is traversed. In particular, for $\tanh(\alpha R_{n-1}/\beta) < R_{n-1}/\beta$ the change in argument $[\arg g]_{C_3} = -\pi$ and for $\tanh(\alpha R_{n-1}/\beta) > R_{n-1}/\beta$ the change in argument is $[\arg g]_{C_3} = \pi$. The signs are reversed because C_3 is traversed in the direction of decreasing v while C_1 is traversed in the direction of increasing v.

On C_2

$$g(w) \sim -\tfrac{1}{2} S\,\mathrm{e}^{S-ix}, \quad \text{as } S \to \infty \tag{26}$$

and it is immediate that $[\arg g]_{C_2} = \pi$ in the limit because x decreases by π as C_2 is traversed. Similarly, as C_4 is traversed $[\arg g]_{C_4} = \pi$.

The calculation of the net change in argument over C and application of the argument principle gives the following results.

THEOREM 2.1. *Let g be the function defined by* (18) *and let $\mathcal{I}_n$ be the condition*

$$\tanh\frac{\alpha R_n}{\beta} < \frac{R_n}{\beta} \tag{27}$$

and $\mathcal{J}_n$ be the condition

$$\tanh\frac{\alpha R_n}{\beta} > \frac{R_n}{\beta}, \tag{28}$$

where $R_n = (2n+1)\pi/2$.

(1) *If either* $\mathcal{I}_{n-1}$ *and* $\mathcal{I}_n$ *or* $\mathcal{J}_{n-1}$ *and* $\mathcal{J}_n$ *are both true then there is exactly one zero of* $g(w)$ *for* $R_{n-1} < \operatorname{Re} w < R_n$.
(2) *If* $\mathcal{I}_{n-1}$ *and* $\mathcal{J}_n$ *are both true, then there are no zeros of* $g(w)$ *for* $R_{n-1} < \operatorname{Re} w < R_n$.
(3) *If* $\mathcal{J}_{n-1}$ *and* $\mathcal{I}_n$ *are both true, then there are exactly two zeros of* $g(w)$ *for* $R_{n-1} < \operatorname{Re} w < R_n$.

A corollary of this result is that, for all $n \geqslant N > 1$ for which $|R_{N-1}| > \beta$, each strip $R_{n-1} < \operatorname{Re} w < R_n$ contains exactly one root. Note that if

$$\tanh \frac{\alpha R_n}{\beta} = \frac{R_n}{\beta} \tag{29}$$

there is a root of (17) with $\operatorname{Re} w = R_n$.

Next C is modified to be the rectangle passing through the four points $(u, v) = (R_n, \pm S)$, $n \geqslant 0$, and $(u, v) = (0, \pm S)$ so that

$$C = C_1 \cup C_2 \cup C_3 \cup C_4, \tag{30}$$

where

$$\begin{aligned} C_1 &= \{u = R_n, v \in (-S, S]\}, \qquad C_2 = \{v = S, u \in [0, R_n)\}, \\ C_3 &= \{u = 0, v \in [-S, S)\}, \qquad C_4 = \{v = -S, u \in (0, R_n]\}, \end{aligned} \tag{31}$$

and again the limit $S \to \infty$ will be taken. The line C_1 is identical to the previous case. On C_2 the asymptotic form (26) applies and $[\arg g]_{C_2} = R_n$; the same result is obtained for C_4. On C_3

$$g(w) = (\alpha + i\beta) \cosh v - v \sinh v \tag{32}$$

and so $\arg g \to \pi (\operatorname{mod} 2\pi)$ as $|v| \to \infty$. Further, $\operatorname{Im} g \neq 0$ for any v so that C_3 does not wind around the origin in the g plane and hence $[\arg g]_{C_3} = 0$. The calculation of the net change in argument over C and application of the argument principle now gives the following results.

THEOREM 2.2. *Let* g *be the function defined by* (18) *and let* $\mathcal{I}_n$ *and* $\mathcal{J}_n$ *be the conditions defined in Theorem* 2.1

(1) *If* $\mathcal{I}_n$ *is true, then there are exactly* $n + 1$ *zeros of* $g(w)$ *for* $0 < \operatorname{Re} w < R_n$.
(2) *If* $\mathcal{J}_n$ *is true, then there are exactly* n *zeros of* $g(w)$ *for* $0 < \operatorname{Re} w < R_n$.

Finally in this section the asymptotic behaviour of the roots is noted. Setting $w = N\pi + W$ in (18), where N is an integer, gives

$$g(w) = (-1)^N \{\Gamma \cos W - (N\pi + W) \sin W\}, \tag{33}$$

so that

$$\lim_{N\to\infty} \frac{g(w)}{N\pi} = (-1)^{N+1} \sin W \tag{34}$$

and asymptotically the roots occur at $W = 0$. Combined with the results of the Theorems 2.1 and 2.2, this shows that the nth root of the dispersion relation with positive real part is asymptotically $w = (n - 1)\pi$ as $n \to \infty$, as is apparent in Figure 1. Thus, the asymptotics of the roots of the dispersion relation are the same for real and complex frequency parameter K.

3. Explicit roots of the dispersion relation

The results obtained in the previous section will not be used to help in the explicit determination of the roots of the dispersion relation (17). The method employs a device first suggested by Burniston and Siewert [8, 9] whereby the zeros of a sectionally analytic function (that is, one with a line of discontinuity) are found by reference to a suitable Riemann–Hilbert problem [7, Sect. 14]. The method was generalised by Anastasselou and Ioakimidis [10] to apply to analytic functions without a discontinuity and both methods were unified by Ioakimidis [11]. Burniston and Siewert [8] obtain explicit expressions for the roots of (17) specifically for the case of real Γ. Their method could be modified to deal with complex Γ but the more straightforward approach of Ioakimidis [11] is adopted here.

First of all the method is outlined briefly in the context of the current problem. Suppose that a function $F(z)$ of a complex variable z is discontinuous for $\operatorname{Re} z \in (-1, 1) \equiv L$, is analytic elsewhere in the z plane and has m zeros in the cut complex plane. The basic aim of the method is to determine a polynomial

$$P_m(z) = \sum_{j=0}^{m} b_j z^j, \tag{35}$$

whose zeros correspond to those of $F(z)$, both in location and multiplicity. In the notation used by Ioakimidis [11], which in turn follows Gakhov [7], the sectionally meromorphic function

$$M(z) = 1/F(z) \tag{36}$$

and the sectionally analytic function

$$\Phi(z) = P_m(z)M(z) \tag{37}$$

are introduced; from their definitions, $M(z)$ and $\Phi(z)$ are discontinuous across L. The key step in the method is to observe that, from its definition, $\Phi(z)$ is a solution to the Riemann–Hilbert problem

$$\Phi^+(t) - G(t)\Phi^-(t) = P_m(t)[M^+(t) - G(t)M^-(t)], \quad -1 < t < 1, \tag{38}$$

where the superscripts $\pm$ denote the boundary values of the function as $z \to t^\pm$, that is as the cut is approached from above $(+)$ and below $(-)$, and $G(t)$ is a function that may be chosen for convenience.

The solution to (38) [7, Sect. 14.5] is

$$\Phi(z) = X(z)[\Psi(z) + Q_p(z)], \tag{39}$$

where $X(z)$ is a particular solution to the homogeneous problem

$$X^+(t) = G(t)X^-(t), \tag{40}$$

$\Psi(z)$ is the sectionally analytic function

$$\Psi(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{P_m(t)[M^+(t) - G(t)M^-(t)]}{(t-z)X^+(t)}\,\mathrm{d}t \tag{41}$$

and $Q_p(z)$ is an arbitrary polynomial of degree p. Two particular choices for $G(t)$ and a corresponding $X(z)$ will be used here. Namely $G(t) = 1$, for which $X(z) = 1$, and $G(t) = -1$ for which

$$X(z) = (z^2 - 1)^{-1/2}, \tag{42}$$

and hence

$$X^+(t) = 1/[i(1-t^2)^{1/2}]. \tag{43}$$

The solution to a Riemann–Hilbert problem is not uniquely determined until the behaviour at infinity of the unknown function is specified precisely; in the present context we implement such a condition by choosing the coefficient b_m of z^m in $P_m(z)$ to be unity.

An expansion about the point at infinity yields

$$\frac{M(z)}{X(z)} = \sum_{j=-\infty}^{l} A_j z^j, \tag{44}$$

and it follows from (37) and (39) that $p = m + l$. If $\Psi(z)$ is also expanded about the point at infinity, then the equality of the expressions for $\Phi(z)$ in (37) and (39) gives

$$\sum_{j=0}^{m} b_j z^j \sum_{j=-\infty}^{l} A_j z^j = -\sum_{k=0}^{\infty}\left[z^{-k-1}\sum_{j=0}^{m} b_j I_{k+j}\right] + Q_{m+l}(z), \tag{45}$$

where

$$I_j = \frac{1}{2\pi i}\int_{-1}^{1}\frac{M^+(t) - G(t)M^-(t)}{X^+(t)} t^j \, \mathrm{d}t. \tag{46}$$

Finally, if the coefficients of z^{-k-1}, $k = 0, \ldots, m-1$, are equated in (45) it is found that

$$\sum_{j=0}^{m}[A_{-k-j-1} + I_{k+j}]b_j = 0, \quad k = 0, \ldots, m-1, \tag{47}$$

which, because the choice $b_m = 1$ has been made, is a set of m linear equations for the m coefficients of $P_m(z)$. This theory is now used to find the roots of the dispersion relation.

After the change of variable

$$w = -i\Gamma z, \tag{48}$$

the dispersion relation (17) may be rewritten as

$$\tan(-i\Gamma z - n\pi) = \frac{1}{iz}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{49}$$

and this may then be rearranged to give

$$F(z) \equiv z + \frac{1}{2\Gamma}\left(\log\frac{z-1}{z+1} - 2n\pi i\right) = 0, \tag{50}$$

where the principle branch of the logarithm is to be taken and the branch cut is chosen to be L, that is the interval $(-1, 1)$ on the real axis. Clearly, $F(z)$ is discontinuous across L and analytic elsewhere in the complex plane and hence its zeros may be obtained by the method described above. In the cut plane

$$-(2n+1)\pi < \mathrm{Im}\left(\log\frac{z-1}{z+1} - 2n\pi i\right) < -(2n-1)\pi \tag{51}$$

and therefore from (48)

$$(2n-1)\tfrac{1}{2}\pi < \mathrm{Re}\, w = u < (2n+1)\tfrac{1}{2}\pi. \tag{52}$$

Thus, choosing different values of n in (50) isolates precisely those strips of the complex w plane covered by the results of Theorem 2.1 and so the number of roots m in the cut plane is known. In particular, m is no greater than two so, at worst, a quadratic equation is solved to determine the zeros of $P_m(z)$.

For this case, the choice $G(t) = -1$ with the corresponding $X^+(t)$ is advantageous in the computation of the integrals I_j as rapid variations in the integrand near $t = \pm 1$ are reduced. Further

$$M^{\pm}(t) = 1/F^{\pm}(t) = \left[t + \frac{1}{2\Gamma}\left(\log\frac{1-t}{1+t} - (2n \mp 1)\pi i\right)\right]^{-1} \tag{53}$$

and, in particular,

$$A_{-1} = \frac{in\pi}{\Gamma}, \quad A_{-2} = \frac{1}{2\Gamma^2}[2\Gamma - \Gamma^2 - 2n^2\pi^2], \quad A_{-3} = -\frac{in\pi}{2\Gamma^3}[2n^2\pi^2 + \Gamma^2 - 4\Gamma], \tag{54–56}$$

$$A_{-4} = \frac{1}{24\Gamma^4}[24n^4\pi^4 + 12n^2\pi^2\Gamma(\Gamma-6) + \Gamma^2(24 - 4\Gamma - 3\Gamma^2)], \tag{57}$$

(these are the only coefficients needed here).

Accurate computation of the integrals is straightforward expect for large $|\Gamma|$ when there can be rapid variations in the integrand near $t = 0$. However, these potential minor inaccuracies are of little consequence because once a good approximation to a root has been obtained it can be refined quite easily by Newton iteration.

4. Double roots of the dispersion relation

Double roots of the dispersion relation occur at $Kh = -kh \tan kh$ whenever kh is a root of (15). It is convenient to write $2kh = W$ so that the location of double roots of (8) is equivalent to looking for the roots of

$$\gamma(W) \equiv \sin W + W = 0. \tag{58}$$

By examining the real and imaginary parts of $\gamma(W)$, we observe that it is easy to see that the origin is the only zero of $\gamma(W)$ that lies on the real or imaginary axes in the complex W plane.

A trivial consequence of the form of $\gamma(W)$ is that if $W = W_0$ is a zero then $-W_0$ and $\pm\overline{W}_0$ are also zeros. Thus, in numerical calculations, it is sufficient to confine attention to one quadrant of the complex $W = U + iV$ plane.

As in Section 2, the argument principle may be applied to obtain further information about the location of the zeros. The details are straightforward so only the results are recorded here. Let C be the rectangle passing through the four points $(U, V) = ((2n - 1/2)\pi, \pm S)$ and $(U, V) = ((2n - 1)\pi, \pm S)$ for any positive integer n, wher the limit $S \to \infty$ is to be taken. Application of the argument principle leads to the following.

THEOREM 4.1. *Let γ be the function defined by* (58). *There are exactly two zeros of $\gamma(W)$ for* $(2n - 1)\pi < \mathrm{Re}\, W < (2n - 1/2)\pi$, *where n is any positive integer. These zeros form a complex conjugate pair.*

A counting result similar to Theorem 2.2 may also be obtained and the result is as follows.

THEOREM 4.2. *Let γ be the function defined by* (58). *There are exactly* $2n$ *zeros of $\gamma(W)$ for* $0 < \mathrm{Re}\, W < (2n - 1/2)\pi$, *where n is any positive integer. These zeros form complex conjugate pairs.*

This last result shows that Theorem 4.1 locates all of the zeros of $\gamma(W)$ in $\mathrm{Re} W > 0$; the zeros in $\mathrm{Re}\, W < 0$ follow by reflection in $\mathrm{Re}\, W = 0$.

The theory outlined in Section 3 will now be used to obtain the zeros of γ explicitly. (Burniston and Siewert [9] give explicit expressions which include the roots of (58) as a special case but again the method of Ioakimidis [11] seems to be more straightforward.) Firstly, (58) may be rewritten as

$$2n\pi - W = \sin^{-1} W = \pi + i \log[(1 - W^2)^{1/2} + iW] \tag{59}$$

for an integer n, where the branch of the inverse sine has been chosen to give

$$\tfrac{1}{2}\pi < \mathrm{Re}\{\sin^{-1} W\} < \frac{3\pi}{2}. \tag{60}$$

With this choice, it follows from (59) that

$$(2n - \tfrac{3}{2})\pi < \mathrm{Re}\, W < (2n - \tfrac{1}{2})\pi, \tag{61}$$

so that each value of n isolates a strip of the W plane for which by Theorems 4.1 and 4.2 there are exactly two complex conjugate roots. In order to apply the results of Section 3, the branch cut is transferred to $L = (-1, 1)$ by the change of variable $W = 1/z$ so that it is required to locate the zeros of

$$F(z) = 1 - z\left\{(2n - 1)\pi - i\log\left[\left(1 - \frac{1}{z^2}\right)^{1/2} + \frac{i}{z}\right]\right\} \tag{52}$$

so that, after careful calculation, as $z \to t^{\pm}$

$$F^{\pm}(t) = 1 + \left\{-(2n - 1)\pi t - \frac{\pi}{2}|t| \pm it\log\left[\left(\frac{1}{t^2} - 1\right)^{1/2} + \frac{1}{|t|}\right]\right\}, \tag{63}$$

Table 1. Values of Γ for which there are double roots of the dispersion relation (18) together with the corresponding roots W of (58).

W	Γ
$4{\cdot}21239 - 2{\cdot}25073i$	$1{\cdot}65061 + 2{\cdot}05998i$
$10{\cdot}7125 - 3{\cdot}10315i$	$2{\cdot}05785 + 5{\cdot}33471i$
$17{\cdot}0734 - 3{\cdot}55109i$	$2{\cdot}27847 + 8{\cdot}52264i$
$23{\cdot}3984 - 3{\cdot}85881i$	$2{\cdot}43112 + 11{\cdot}6888i$
$29{\cdot}7081 - 4{\cdot}09370i$	$2{\cdot}54799 + 14{\cdot}8458i$

[see 9, Eqn. 2.1 with k even]. Slightly more accurate results are obtained for $G(t) = 1$ (rather than $G(t) = -1$) and the required expansion coefficients are

$$A_{-1} = -\frac{1}{(2n-1)\pi}, \qquad A_{-2} = A_{-3} = 0, \qquad A_{-4} = \frac{1}{6(2n-1)^2\pi^2}. \tag{64}$$

The five roots of (58) within the fourth quadrant and with smallest modulus are given in Table 1, along with the corresponding values of Γ.

5. The expansion theorem

Attention is now turned to the structure of the eigenfunction expansions for the problem where the frequency parameter in (8) is complex. The standard expansion theorems found in most text books do not apply as the vertical eigenfunction problem is no longer self-adjoint. Other problems where the standard theory does not apply arise, for example, in underwater acoustics [12] and heat conduction [13].

An operator T is self-adjoint if, for all suitable functions u and v

$$\langle Tu, v\rangle = \langle u, Tv\rangle, \tag{65}$$

(the inner product is defined in equation 12). Integration by parts show that the operator defined by (4–6) is self-adjoint if and only if K is real. Hence the porous structure problem, where K is complex, is not self-adjoint and the expansion theorem (14) does not apply.

The expansion theorem for non-self-adjoint ordinary differential operators is discussed by Coddington and Levinson [5, Chap. 12] and the theory described there is now applied to the present problem. The eigenvalues of the problem (4–6) are given by $\lambda = k^2$, where k is a, now complex, root of the dispersion relation (8). Let C_n be a closed contour in the complex λ plane which encircles in an anticlockwise direction the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, arranged in order of increasing modulus and with repetitions included. The general expansion theorem [5, p. 299] says that, for suitable functions f

$$f(y) = -\lim_{n\to\infty}\int_{-h}^{0} P_n(y, \eta) f(\eta)\,\mathrm{d}\eta, \tag{66}$$

where

$$P_n(y,\eta) = \frac{1}{2\pi i}\int_{C_n} G(y,\eta;\lambda)\,\mathrm{d}\lambda. \tag{67}$$

G now denotes the Green's function for the particular problem under consideration, and provided suitable convergence properties can be established for the particular problem under consideration. Fortunately, the problem here falls into a class investigated in detail by Coddington and Levinson [5, Thm. 2.1, p. 303] and the expansion theorem (66) is indeed applicable.

The Green's function $G(y,\eta;\lambda)$ for the problem (4–6) is required to satisfy the additional condition

$$\lim_{\delta\to 0^+}\left\{\frac{\partial G}{\partial y}(\eta+\delta,\eta;\lambda) - \frac{\partial G}{\partial y}(\eta-\delta,\eta;\lambda)\right\} = -1 \tag{68}$$

and it is readily found that

$$G(y,\eta;\lambda) = -\frac{(k\cos ky_> + K\sin ky_>)\cos k(y_< + h)}{k(K\cos kh + k\sin kh)}, \quad k = \lambda^{1/2}, \tag{69}$$

where

$$y_< = \min(y,\eta) \quad \text{and} \quad y_> = \max(y,\eta). \tag{70}$$

This Green's function has poles at values of λ corresponding to the roots of the dispersion relation (8) so that, by the residue theorem

$$P_n(y,\eta) = \sum_{m=1}^{n} R_m(y,\eta), \tag{71}$$

where R_m is the residue of G at $\lambda = \lambda_m$. If the eigenvalues are known then the residues at the poles of the Green's function can be calculated and the form of the general expansion found. As has been observed previously in this paper, at isolated values of K there is a double root of the disperison relation and hence a double pole of the Green's function. For almost all values of K the Green's function has only simple poles.

The residue of the Green's function for a pole of order p_m at $\lambda = \lambda_m$ is readily evaluated by standard means and may be written

$$R_m(y,\eta) = -\sum_{q=1}^{p_m} \overline{\psi}_{m,p_m-q+1}(\eta)\chi_{m,q}(y). \tag{72}$$

The set $\{\chi_{m,q}\}$ are the so-called 'generalised eigenfunctions' of the problem (4–6), while the set $\{\psi_{m,q}\}$ are the generalised eigenfunctions of the adjoint problem. For the case of a simple pole, $p_m = 1$,

$$\chi_{m,1} = \frac{\cos k_m(y+h)}{N_m} \quad \text{and} \quad \psi_{m,1} = \overline{\chi}_{m,1} \tag{73}$$

and they satisfy

$$\langle \chi_{m,1}, \psi_{m,1} \rangle = 1. \tag{74}$$

For the case of a double pole $p_m = 2$, the generalised eigenfunctions are

$$\chi_{m,1} = -\frac{2\cos k_m(y+h)}{\cos^2 k_m h} \quad \text{and} \quad \psi_{m,1} = \overline{\chi}_{m,1}, \tag{75}$$

$$\begin{aligned} \chi_{m,2} &= \tfrac{1}{6}(4\sin^2 kh - 3)\cos k_m(y+h) + k_m(y+h)\sin k_m(y+h) \quad \text{and} \\ \psi_{m,2} &= \overline{\chi}_{m,2}, \end{aligned} \tag{76}$$

with

$$\langle \chi_{m,1}, \psi_{m,1} \rangle = \langle \chi_{m,2}, \psi_{m,2} \rangle = 0 \quad \text{and} \quad \langle \chi_{m,1}, \psi_{m,2} \rangle = \langle \chi_{m,2}, \psi_{m,1} \rangle = 1. \tag{77}$$

In the double-pole case, although the residue is well defined, there is a degree of arbitrariness in how the residue is split to form the generalised eigenfunctions $\{\chi_{m,q}, \psi_{m,q}; q = 1, 2\}$; the functions given here are chosen specifically to satisfy (77). Generalised eigenfunctions corresponding to different eigenvalues are biorthogonal so that

$$\langle \chi_{m,q}, \psi_{n,r} \rangle = 0, \qquad m \neq n. \tag{78}$$

Superficially this result and (74) are identical to the orthogonality condition (13). However, because of the definition of the inner product (12), when K is complex it is not technically correct to describe the functions $\{\chi_{m,1}, m = 1, 2, \ldots\}$ as forming an orthonormal set even when there are no double poles. Rather, these functions are biorthogonal to the eigenfunctions $\{\psi_{m,1}, m = 1, 2, \ldots\}$ of the adjoint problem.

With the above definitions, the general expansion theorem is

$$f = \sum_{m=1}^{\infty} \sum_{q=1}^{p_m} \langle f, \psi_{m,p_m-q+1} \rangle \chi_{m,q}. \tag{79}$$

As noted at the end of Section 2, the asymptotic behaviour of the roots of the dispersion relation are the same for real and complex K. Thus, expansions of the form (79) in a porous-structure problem will display the same convergence properties as similar expansions in standard water-wave problems. For real K, all poles of the Green's function are simple and $\psi_{m,1} = \chi_{m,1} \equiv \chi_{m-1}$, as defined in equation (9), so that (79) reduces to (14) after a suitable relabelling of the eigenvalues.

6. Solutions of Laplace's equation

The expansion theorem (79) may be used to find solutions of water wave problems. For example, suppose that a two- dimensional solution $\phi(x, y)$ of Laplace's equation is required satisfying the boundary conditions

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{on} \quad y = -h \quad \text{and} \quad \frac{\partial \phi}{\partial y} = K\phi \quad \text{on} \quad y = 0. \tag{80}$$

From (79), the solution is sought in the form

$$\phi(x, y) = \sum_{m=1}^{\infty} \sum_{q=1}^{p_m} C_{m,q}(x)\chi_{m,q}(y), \tag{81}$$

which satisfies the Laplace equation provided

$$\sum_{m=1}^{\infty} \sum_{q=1}^{p_m} \{C''_{m,q}(x)\chi_{m,q}(y) + C_{m,q}(x)\chi''_{m,q}(y)\} = 0. \tag{82}$$

Now by differentiation

$$\chi''_{m,1} = -k_m^2 \chi_{m,1} \quad \text{and} \quad \chi''_{m,2} = -k_m^2 \chi_{m,2} - k^2 \cos^2 kh\, \chi_{m,1}, \tag{83}$$

so that (78) may be used to isolate terms corresponding to distinct eigenvalues. For a simple pole this results in

$$C''_{m,1} - k_m^2 C_{m,1} = 0, \tag{84}$$

which has the general solution

$$C_{m,1}(x) = a_m\, \mathrm{e}^{k_m x} + b_m\, \mathrm{e}^{-k_m x}, \tag{85}$$

where a_m and b_m are arbitrary constants. For a double pole, application of (78) and the biorthogonality properties (77) yields

$$C''_{m,2} - k_m^2 C_{m,2} = 0 \quad \text{and} \quad C''_{m,1} - k_m^2 C_{m,1} = k^2 \cos^2 kh C_{m,2}, \tag{86}$$

which have general solutions

$$C_{m,2}(x) = c_m\, \mathrm{e}^{k_m x} + d_m\, \mathrm{e}^{-k_m x} \tag{87}$$

and

$$C_{m,1}(x) = a_m\, \mathrm{e}^{k_m x} + b_m\, \mathrm{e}^{-k_m x} + \tfrac{1}{2} kx \cos^2 kh (c_m\, \mathrm{e}^{k_m x} - d_m\, \mathrm{e}^{-k_m x}), \tag{88}$$

where c_m and d_m are further arbitrary constants. This gives the 'missing' eigenfunctions and a simple rearrangement gives agreement with the calculation of Dalrymple *et al*. [3].

7. Conclusion

The construction of eigenfunction exapnsions within porous structures has been examined in detail and a number of results have been used to clarify the construction of such expansions. The method of Ioakimidis [11] has been applied to the dispersion relation and this allows straightforward computation of its roots throughout the complex plane. The theory of non-self-adjoint differential operators [5] has been used to show how the formal construction of eigenfunction expansions can be carried out.

A further development of the model discussed here to waves within a layered porous structure has been described by Yu and Chwang [14]. This result in a much more complicated dispersion relation and it is not clear that the methods used in this paper can be easily applied to locate the roots. Neither is it clear, without further work, whether or not the convergence results of Coddington and Levinson [5] allow the formal expansion theorem to be applied to this case.

Acknowledgment

The author is grateful to Dr Mark Groves for sharing his knowledge of non-self-adjoint operators.

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